COORDINATED SCIENCE LABORATORY

TABLES OF HERTZIAN CONTACT-STRESS COEFFICIENTS

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Tables of Hertzian Contact-Stress Coefficients

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Abstract

Formulations are given for the coefficients λ, μ, ν defined by H. Hertz in terms of the solution of a transcendental equation involving elliptic integrals and used by him to describe the deformation of bodies subjected to contact stresses. Methods of approximate calculation are explained, and errors in the tables prepared by Hertz are noted. For the purpose of providing a more extensive and more accurate tabulation, using an automatic digital computer, these coefficients are reformulated so that a large part of the variation is expressed by means of easily-interpreted elementary formulae. The remainder of the variation is tabulated to 6 places for 100 values of the argument. Graphs of the coefficients are also provided.

Introduction

In two papers [1] published in 1881 and 1882, H. Hertz reported his analysis describing the elastic stress system generated in two bodies initially making a frictionless contact at a single point upon being pressed together with a force F. The description includes formulae for the overall deformation δ, the distance through which parts of the bodies remote from the contact approach one another, and the semi major and minor axes, a and b, of the ellipse bounding the contact interface. These formulae involve elementary algebraic expressions multiplied by certain coefficients [2], λ, μ, ν, for each of δ, a, b, respectively, which coefficients are defined

in terms of a solution to a transcendental equation involving elliptic integrals. Rather then give detailed prescriptions for the calculation of these coefficients, Hertz tabulated them, saving his readers from a "wearisome" task, and leaving the main body of his papers uncluttered with a digression on such matters.

Unfortunately, the tables prepared by Hertz were given to a rather coarse interval for the argument, and the values of the coefficients were specified to only 4 decimals (later rounded to 3). These tables served the purpose, at least, of providing explicit expressions, and useful ones, presumably with interpolation, for describing many practical cases in which very precise estimates would not be needed. Later versions of the table of Hertz have been published [3,4] in which interpolated values for half of the original intervals appear. Unfortunately also, however, errors which appear in the table of Hertz are reproduced in these later versions, so that, for certain entries, the values are not accurate to 3 decimals nor yet to the 4 decimals originally quoted.

The discovery of these errors serves as the motivation for an examination of procedures to be used in making fresh calculations of the Hertzian coefficients. It is readily believed that finding the solution to the transcendental equation of Hertz by numerical methods, and the evaluation of elliptical integrals using that solution as an argument, involve interpolations that would be exceedingly wearisome to implement by hand methods to any reasonable degree of accuracy. Such procedures are best left to an automatic digital computer. The overall plan of the calculation, however, is one in which the solution of the transcendental equation appears as a

parameter which is to be eliminated from expressions connecting the coefficients with a certain variable τ, to be, in the end, regarded as the independent variable. Both the coefficients and τ are given as explicit functions of that parameter. This observation leads to the suggestion that curves of λ, μ, ν may be plotted parametrically versus τ to achieve a graphical elimination of the parameter, essentially replacing numerical interpolations by graphical ones. The accuracy with which this can be done may be enhanced by the use of certain simple approximations λ_o, μ_o, ν_o such that the ratios λ/λ_o, μ/μ_o, ν/ν_o exhibit small variations. For smaller values of τ, where these ratios show larger variations, linear interpolation on logarithmic scales is seen to be feasible, even by manual numerical methods. An accuracy of 0.1% seems to be readily accessible by such methods.

Despite the feasibility of graphical methods, one could desire access to an accurate table, and there is little reason for such a table not to exist. Accordingly, the tabulation has been done using a CDC-1604 computer. In designing the table, the principle was followed of seeking a formulation of the coefficients in which a large part of the variation could be expressed by means of elementary functions of rather simple structure that would exhibit a straightforward relevance to the contact-stress problem. Such a formulation was found, and the tabulated part is of the reformulated coefficients denoted by λ^*, μ^*, ν^*. As a further convenience, the tabulation is given for the argument $t = \cos\tau$.

Hertzian Deformation Formulae

Hertz took the initial shapes of the bodies to be pressed together as being described by quadric surfaces, as elliptic or hyperbolic paraboloids,

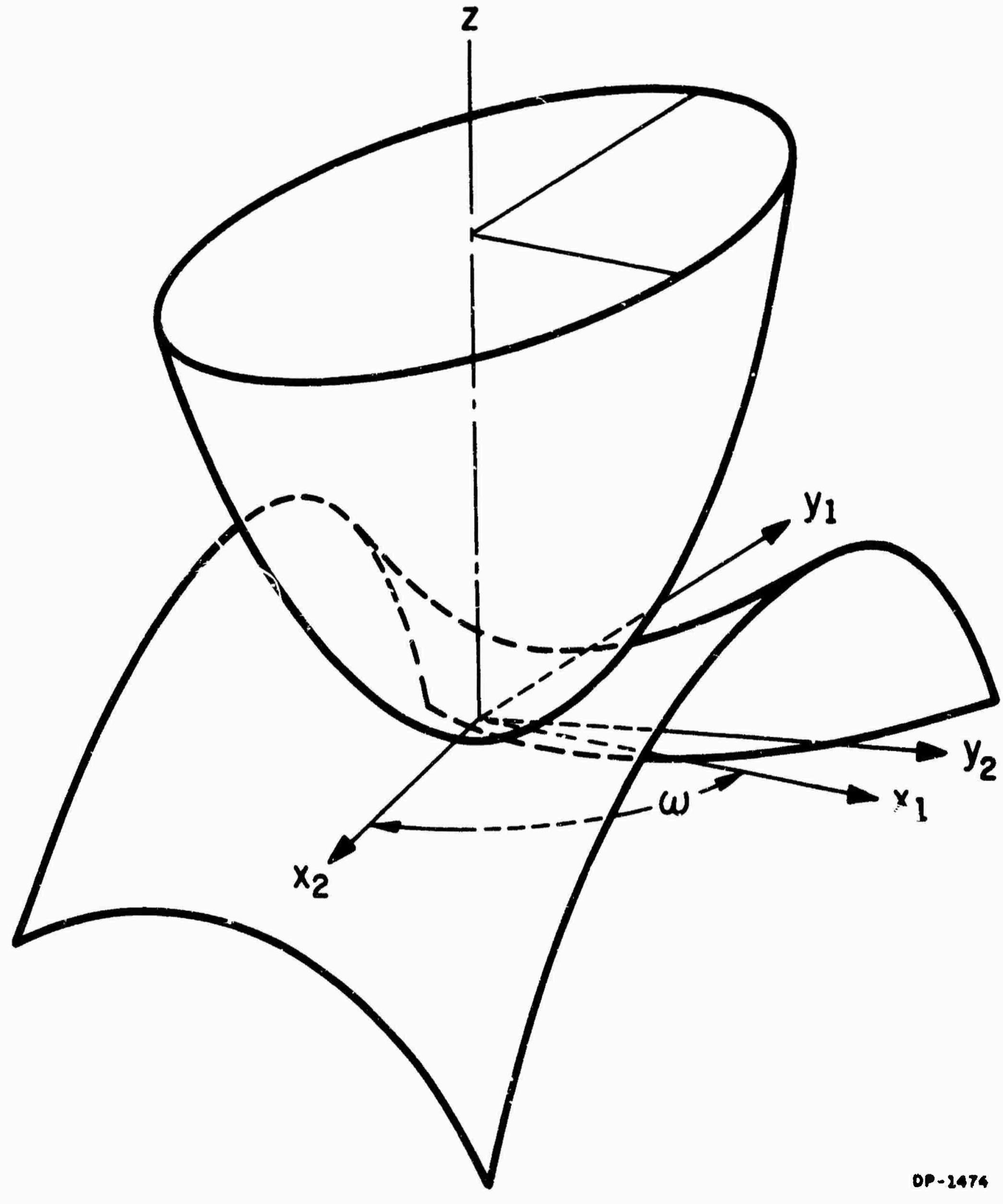

Fig. 1. Geometry of the contact between quadric surfaces illustrated by elliptic and hyperbolic paraboloids for the upper (principal axes x_1, y_1, z) and lower (principal axes x_2, y_2, z) surfaces, respectively, in which the axes for one surface have been turned through an angle ω relative to those for the second.

in the neighborhood of the initial point of contact. The contact was assumed to be frictionless, so that the force F with which the bodies were to be pressed together would necessarily be directed along the normal to the initial tangent plane, and the resulting contact interface would be free of traction [5]. He regarded his description of the initial shapes as being applicable to a greater variety of shapes, applicable to the contact of a sphere with a cylinder, for example, because the infinitesimal dimensions of the contact interface, small in comparison to the initial radii of curvatures of the bodies, was necessitated, for practical materials, to maintain the validity of the assumption of Hooke's law. It will be recognized, however, that there are exceptions, the contact of a sphere with the interior of a closely-fitting cylinder, for example, in which the contact will not be infinitesimal, even for stress regimes for which Hooke's law will remain valid. For these, the quadric-surface approximation may be of questionable utility.

The undeformed quadric surfaces are completely specified by the principal-axis curvatures at the initial point of contact γ_{11} and γ_{12} for the first body, and γ_{21} and γ_{22} for the second body. These curvatures are reciprocals of the corresponding radii of curvature, and each is taken to be positive if the center of curvature for each instance lies within the body. The principal axes of curvature for one body are taken to make an angle ω with those for the other body. See Fig. 1. The deformation formulae are expressible in terms of certain combinations of these principal-axis curvatures. These combinations are

$$\gamma_3 = [(\gamma_{11}+\gamma_{21})(\gamma_{12}+\gamma_{22})+(\gamma_{11}-\gamma_{12})(\gamma_{21}-\gamma_{22})\sin^2\omega]^{\frac{1}{2}}, \tag{1a}$$

$$\gamma_4 = \tfrac{1}{2}[(\gamma_{11}-\gamma_{12})^2+(\gamma_{21}-\gamma_{22})^2+2(\gamma_{11}-\gamma_{12})(\gamma_{21}-\gamma_{22})\cos 2\omega]^{\frac{1}{2}}, \tag{1b}$$

$$\gamma_5 = \tfrac{1}{2}(\gamma_{11}+\gamma_{12})+\tfrac{1}{2}(\gamma_{21}+\gamma_{22}), \tag{1c}$$

and it may be seen that they are related as the sides of a right triangle of which γ_5 would be the hypotenuse, and of which the auxiliary angle

$$\tau = \cos^{-1}(\gamma_4/\gamma_5) \tag{1d}$$

would be the angle opposite γ_3. In the case of contact between identical cylinders, the auxiliary angle has a simple interpretation; it is then the same as ω.

It seems appropriate to refer to the particular combination of elastic constants appearing in the deformation formulae as the Hertzian modulus. It is [3]

$$H = (4/3)E/(1-n^2), \tag{2}$$

in which E is Young's modulus and n is Poisson's ratio. This is a stiffness modulus; for the two bodies jointly, the compliance moduli would be additive, so that the joint stiffness modulus would be

$$H_{12} = (H_1^{-1}+H_2^{-1})^{-1}, \tag{3}$$

in which the subscripts refer to first and second bodies. In the formulae that follow, H_{12} will be written simply as H with the understanding that the quantity given by Eq. (3) is meant.

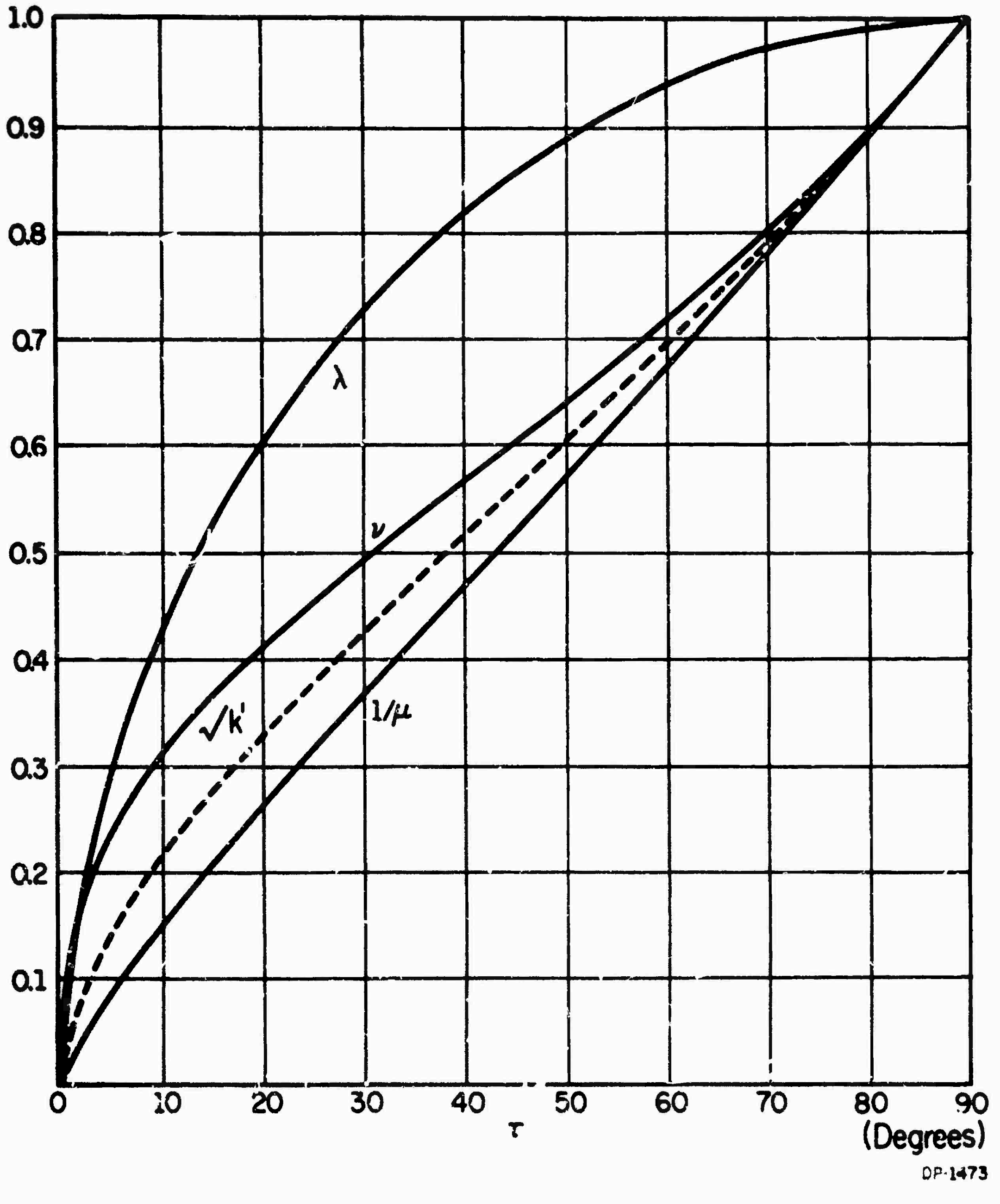

Fig. 2. Plots of Hertzian contact-stress coefficients, $\lambda, 1/\mu, \nu$, vs the auxiliary angle τ, together with the complementary elliptic-integral modulus $k' = \nu/\mu$. The complementary modulus (shown in square root) was used as a plotting parameter, via Eqs. (8). These coefficients define the overall deformation and the dimensions of the interface ellipse, via Eqs. (4), in their dependence on the undeformed-body curvature parameters of Eqs. (1).

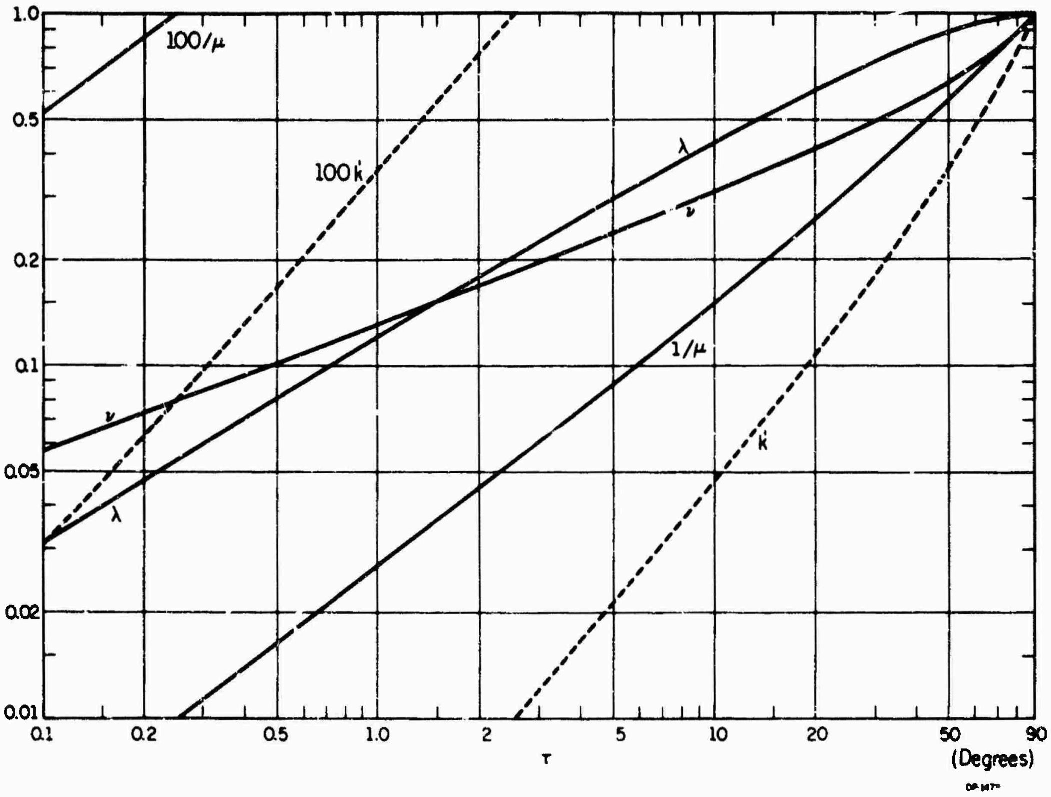

Fig. 3. Plots of the same coefficients as in Fig. 2, except to logarithmic scales. The straightness of the curves for τ less than about 20° indicates that logarithmic interpolation can be more accurate than the linear, for such values of τ.

The three deformation formulae of Hertz may be written

$$\gamma_5\delta = (F\gamma_5^{\,2}/H)^{2/3}\lambda, \tag{4a}$$

$$\gamma_5 a = (F\gamma_5^{\,2}/H)^{1/3}\mu, \tag{4b}$$

$$\gamma_5 b = (F\gamma_5^{\,2}/H)^{1/3}\nu, \tag{4c}$$

in which the deformation δ is the distance through which the parts of the two bodies remote from the contact approach one another, and a and b are the semimajor and semiminor axes of the ellipse that is the projection upon the initial tangent plane of the boundary of the contact interface. The interface itself is also a quadric surface. A plot of the interface pressure would again be a quadric surface, with maximum pressure in the center, a maximum which is 50% greater than the average pressure. The ratio of that average P_{av} to the Hertzian modulus may be computed by dividing $F\gamma_5^{\,2}/H$ by $\gamma_5^{\,2}\pi ab$ to obtain

$$P_{av}/H = (F\gamma_5^{\,2}/H)^{1/3}\mu\nu\pi \tag{5}$$

In Eqs. (4) and (5), the Hertzian coefficients λ, μ, ν are expressible in terms of complete elliptic integrals of modulus k, which modulus is to be found as the solution of a transcendental equation involving the auxiliary angle τ and these same elliptic integrals. Plots of λ, μ, ν and the complementary modulus

$$k' = (1-k^2)^{\frac{1}{2}}, \tag{6}$$

or its square root, are given in Figs. 2 and 3. Formulae for these are

given below.

Formulae for the Hertzian Coefficients

With the help of the Byrd-and-Friedman handbook [6], the elliptic integrals set down by Hertz may be cast into the standard forms

$$(1/a^3k^2)(K-E) = \tfrac{1}{2}\pi\gamma_5(H/F)\sin^2\tau/2, \tag{7a}$$

$$(1/a^3k^2k'^2)(E-k'^2K) = \tfrac{1}{2}\pi\gamma_5(H/F)\cos^2\tau/2, \tag{7b}$$

in which the complementary modulus k' for the complete elliptic integrals K(k) and E(k), of first and second kinds, respectively, is given by

$$k' = b/a = \nu/\mu = (1-k^2)^{\frac{1}{2}}. \tag{7c}$$

(The complementary modulus was written by Hertz without the accent, as simply k, since he elected to omit the reduction to normal Legendre notation used here, and thus found no occasion to refer to the modulus itself, for which the unaccented symbol is usually reserved. Also, in his article of 1881, misprints sometimes give the appearance of an interchange of the roles of a and b in the formulae. These misprints are corrected in his 1882 article.) Equations (7) may be regarded as determining μ and ν, upon the elimination between them of the modulus and its complement, as functions of the auxiliary angle τ.

Similarly, the elliptic integral expressing the deformation δ (for which Hertz used the same symbol, except in bold face, as for the semi-major axis of the interface ellipse) may be cast into normal form, resulting in the formula

$$\lambda\nu = (2/\pi)k'K. \tag{8a}$$

The elimination of the modulus is to be obtained by solving the transcendental equation obtained by dividing Eq. (7a) by Eq. (7b) giving an expression for $\tan^2\tau/2$ in terms of the modulus regarded as the unknown. After some trigonometric manipulation, this transcendental equation may be written as

$$(k'^2/k^2)(K-E)/E = \sin^2\tau/2, \tag{8b}$$

and a complementary expression may be obtained for $\cos^2\tau/2$, suitable for insertion into Eqs. (7a) and (7b). When this insertion is made, the definitions given by Eqs. (4) and (7c) may be used to write

$$\mu^3 = (2/\pi k'^2)E, \tag{8c}$$

$$\nu^3 = (2k'/\pi)E, \tag{8d}$$

after some manipulation. The solution of Eq. (8b) provides the value of k' and, via Eq. (7c), the value of k to be inserted in the remaining Eqs. (8) for the calculation of λ, μ, ν as functions of τ.

<u>Manual Computation of the Coefficients</u>

Obtaining the explicit solution of the transcendental equation of Hertz, Eq. (8b) or its equivalent, is indeed "wearisome" if numerical methods executed "by hand" are to be used to obtain results accurate to 3 or 4 decimals, or better. The principal difficulty obtains in assuring that interpolations of sufficient accuracy may be made in the readily-available tables of K and E, especially for modulus values near unity where K has a logarithmic singularity. As may be seen from Figs. 2 and 3 this obtains for τ near 0 (k' near 0).

For cases in which a limited accuracy would be satisfactory, values may be read directly from Figs. 2 and 3, of course, or interpolations may be made in the table of Hertz to obtain the values of μ and ν, and thus the values of k' may be obtained for use in Eq. (8a). The table of Hertz is, except for three instances, accurate to within one unit in the fourth decimal. For the three exceptions, at $\tau = 10^{\circ}$, 20°, and 70°, corrections are given here in Table I. These errors in the table of Hertz were

Table I. Corrections verified by automatic digital computer of the erroneous values for μ, ν appearing in the table of Hertz [1].

τ	Source	μ	ν
10°	Computer	6.6115	0.3110
	Hertz	6.6120	0.3186
20°	Computer	3.8160	0.4121
	Hertz	3.7799	0.4079
70°	Computer	1.2851	0.7999
	Hertz	1.2835	0.8017

discovered by graphical means and verified by computations on an automatic digital computer to a much higher accuracy. Hertz did not tabulate values of λ. His errors are reproduced in later tables [3,4] of the Hertzian coefficients. As may be seen from Fig. 2, interpolation in a table for μ will give the more accurate results if based on $1/\mu$, if τ not be too small. There is, however, a singularity in slope in each of λ, μ, ν, and $\sqrt{k'}$ at $\tau = 0$. Similarly, linear interpolations in ν will not be quite so accurate, since the curvature may be observed to "infect" a larger range of values near $\tau = 0$, and the same problem may also be observed for λ. In these

cases, interpolation on logarithmic scales will be more accurate, as may be seen from Fig. 3.

An explicit solution of the transcendental equation of Hertz may be avoided, along with interpolations directly in tables of K and E. Instead, one may select, with the help of Figs. 2 and 3, values of k' for which tabular values of K and E are available, and which correspond to τ values bracketing the desired τ value. Then for these k' values, the corresponding values of λ, μ, ν, and τ are computed from Eqs. (8). If τ be small, interpolation in these values to logarithmic scales, either by plotting λ, μ, ν vs τ on log-log paper, or numerically, will yield the λ, μ, ν values for the desired value of τ. For these small values of τ, the approximation

$$K \approx \ln(4/k'), \tag{9}$$

since k' will also be small, will sometimes be helpful. If τ not be small, another interpolation aid will be needed.

For these larger values of τ, it is helpful to compute the ratios λ/λ_o, μ/μ_o, ν/ν_o, in which λ_o, μ_o, ν_o are approximations given by

$$\lambda_o = (\sin\tau)^{\frac{1}{2}}, \tag{10a}$$

$$\mu_o = (1/\sqrt{2})/\sin(\tau/2), \tag{10b}$$

$$\nu_o = (1/\sqrt{2})\lambda_o/\cos(\tau/2), \tag{10c}$$

obeying

$$\lambda_o\mu_o\nu_o = 1, \tag{10d}$$

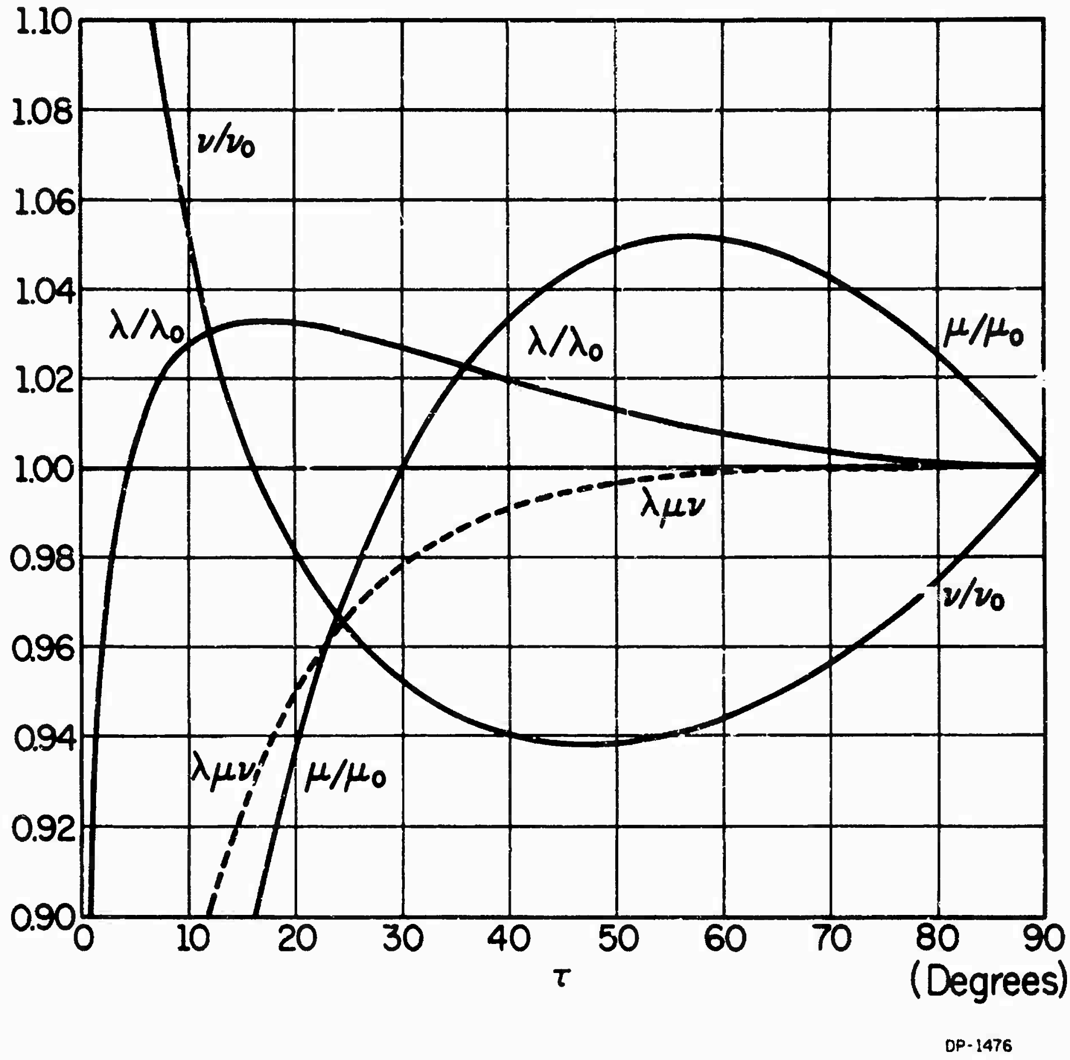

Fig. 4. Plots of Hertzian coefficients as ratios to certain approximating functions serving as interpolating aids. Graphs such as these may be readily made to be read to an accuracy of 0.1% as parametric plots <u>via</u> Eqs. (8) and (10).

also approximately obeyed by $\lambda\mu\nu$. The reason for computing these ratios is that they exhibit a range of variation, for τ not too small, that is rather slight. Thus, interpolation errors stemming from the neglect of curvature (second-order differences) will be lessened in seriousness, or may easily be corrected by computing values for 3 or more values of τ neighboring the desired one. Graphs like those shown in Fig. 4 may be prepared that may be read to an accuracy of 0.1% without excessive difficulty. The noted errors in the table of Hertz are particularly conspicuous on such graphs.

Automatic Digital Computation of the Coefficients

In the interest of preparing a table to small increments of τ and great accuracy in λ, μ, ν a program was written for an automatic digital computer (Control Data Corporation model CDC-1604). This program used an elliptic-integral subroutine based on an approximation accurate to within about 1.5×10^{-8}, as described by Hastings [7]. The approximate elliptic integrals are represented by expressions of the form

$$P(\eta)+Q(\eta)\ln\eta, \tag{11}$$

in which P and Q are polynomials for which appropriate coefficient values have been tabulated by Hastings, and η is the square of the complementary modulus defined in Eq. (6).

Equations (8) were recast in terms of η and a derivative expression for Eq. (8b) was obtained with the help of well-known formulae [6]. Based on these formulations, a Newton-Raphson routine was written

to solve Eq. (8b) for η to an accuracy of 5 parts in 10^{10}, starting from the approximation

$$\eta_o = (\tau/90^o)^3. \tag{12}$$

From 3 to 5 iterations sufficed to solve Eq. (8b) for τ values in the range from 0.1^o to 89^o. The values of η so obtained were then used to compute λ, μ, ν with an overall error primarily imposed by the subroutines and believed to be less than about 2×10^{-8}. Tables of these values are given to 6-place accuracy in the appendix.

Reformulation of the Coefficients

In planning for the publication of a table of Hertzian coefficients, the experience obtained in preparing Fig. 4 led to the thought that a formulation might be found for which the tabulated values would encompass a rather small range of variation for nearly all of the argument values. If so, the accuracy with which a table of fixed size could be used would thus be enhanced. At the same time, however, it was desired that any reformulation not require extensive additional computations in order for the tabulated values to be related to the contact-stress situation.

One possible reformulation seeks to relate the semiaxes of the elliptical boundary of the interface with the semiaxes of the curve of intersection that would obtain if the two surfaces simply interpenetrated one another without deformation, since Hertz had already noted that the dimensions of these two ellipses were nearly the same.

Let the interpenetration distance be called d. If one surface

be defined by the equation

$$z = A_1x^2+Cxy+B_1y^2, \tag{13a}$$

while the other be defined by

$$z-d = A_2x^2+Cxy+B_2y^2, \tag{13b}$$

then the projection upon the x,y plane of their intersection is the curve given by

$$d = Ax^2+By^2, \tag{13c}$$

in which $A = A_1-A_2$ and $B = B_1-B_2$. As Hertz pointed out, it is always possible to choose a coordinate system in which the coefficients of the xy term in Eqs. (13a) and (13b) are one and the same while A and B are each positive. After some algebraic manipulation, it is possible to obtain the expressions of Hertz for γ_5 and γ_4:

$$\gamma_5 = A+B, \tag{14a}$$

$$-\gamma_4 = A-B, \tag{14b}$$

and thus obtain explicit expressions for A and B:

$$A = \tfrac{1}{2}(\gamma_5-\gamma_4) = \gamma_5\sin^2\tau/2, \tag{15a}$$

$$B = \tfrac{1}{2}(\gamma_5+\gamma_4) = \gamma_5\cos^2\tau/2. \tag{15b}$$

Thus, it is seen that the ellipse of interpenetration given by Eq. (13c) has the semiaxes

$$\alpha = [2d/(\gamma_5 - \gamma_4)]^{\frac{1}{2}}, \tag{16a}$$

$$\beta = [2d/(\gamma_5 + \gamma_4)]^{\frac{1}{2}}. \tag{16b}$$

It is the casting of Eqs. (4b) and (4c) into this form that is the basis for the reformulation.

It may be seen from Eqs. (15) that Eqs. (16) may be written

$$\alpha = (2d/\gamma_5)^{\frac{1}{2}}\mu_o, \tag{17a}$$

$$\beta = (2d/\gamma_5)^{\frac{1}{2}}\nu_o/\lambda_o, \tag{17b}$$

using the definitions given in Eqs. (10), whereas from Eqs. (4) there obtain

$$a = (\delta/\gamma_5)^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\mu, \tag{18a}$$

$$b = (\delta/\gamma_5)^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\nu, \tag{18b}$$

or

$$a = (\delta/\gamma_5)^{\frac{1}{2}}\mu_o\bar{\mu}, \tag{19a}$$

$$b = (\delta/\gamma_5)^{\frac{1}{2}}(\nu_o/\lambda_o)\bar{\nu}, \tag{19b}$$

in which

$$\bar{\mu} = (\mu/\mu_o)(\lambda/\lambda_o)^{-\frac{1}{2}}\lambda_o^{-\frac{1}{2}}, \tag{20a}$$

$$\bar{\nu} = (\nu/\nu_o)(\lambda/\lambda_o)^{-\frac{1}{2}}\lambda_o^{\frac{1}{2}} \tag{20b}$$

represent possible reformulations.

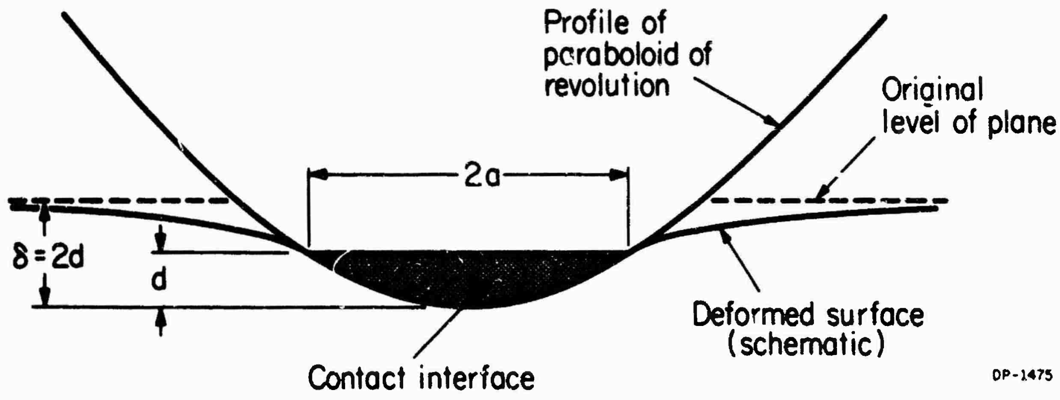

Fig. 5. Schematic drawing of the indentation of a compliant plane on the part of a stiff "sphere," a paraboloid of revolution. The overall deformation δ is twice the interpenetration distance d.

In terms of these reformulated coefficients, the expressions analogous to Eqs. (16) are

$$a = [\delta/(\gamma_5-\gamma_4)]^{\frac{1}{2}}\bar{\mu}, \tag{21a}$$

$$b = [\delta/(\gamma_5+\gamma_4)]^{\frac{1}{2}}\bar{\nu}. \tag{21b}$$

Since it may be seen from Eqs. (20) and Fig. 4 that $\bar{\mu}$ and $\bar{\nu}$ will depart very little from unity for τ not too small (γ_4 not too large in comparison to γ_5), the principal difference between Eqs. (21) and Eqs. (16) is at .butable to the factor 2 multiplying d. It may be said that, for interpenetration and interface ellipses of the same size and not too elongated, the deformation δ is about twice the interpenetration d. This observation is exact for circular interfaces for which $\gamma_4 = 0$ and $\bar{\mu} = \bar{\nu} = 1$. This case is illustrated in Fig. 5, depicting a stiff "sphere" in contact with a relatively compliant plane. It may also be observed with the help of Fig. 6 that, as Hertz had noted, the interface ellipse is relatively longer and narrower than the interpenetration ellipse would have been.

While the reformulation of μ and ν appears to offer some interpretive advantages, besides providing, as shown in Fig. 6, a reduction in the range of variation, the reformulation for λ does not do quite so much. It does at least reduce the range of variation while offering no interpretive difficulties; it consists in the substitution of γ_3 for γ_5 via the triangle relationship noted in connection with Eqs. (1):

$$\gamma_3 = \gamma_5\lambda_o^{2}. \tag{22}$$

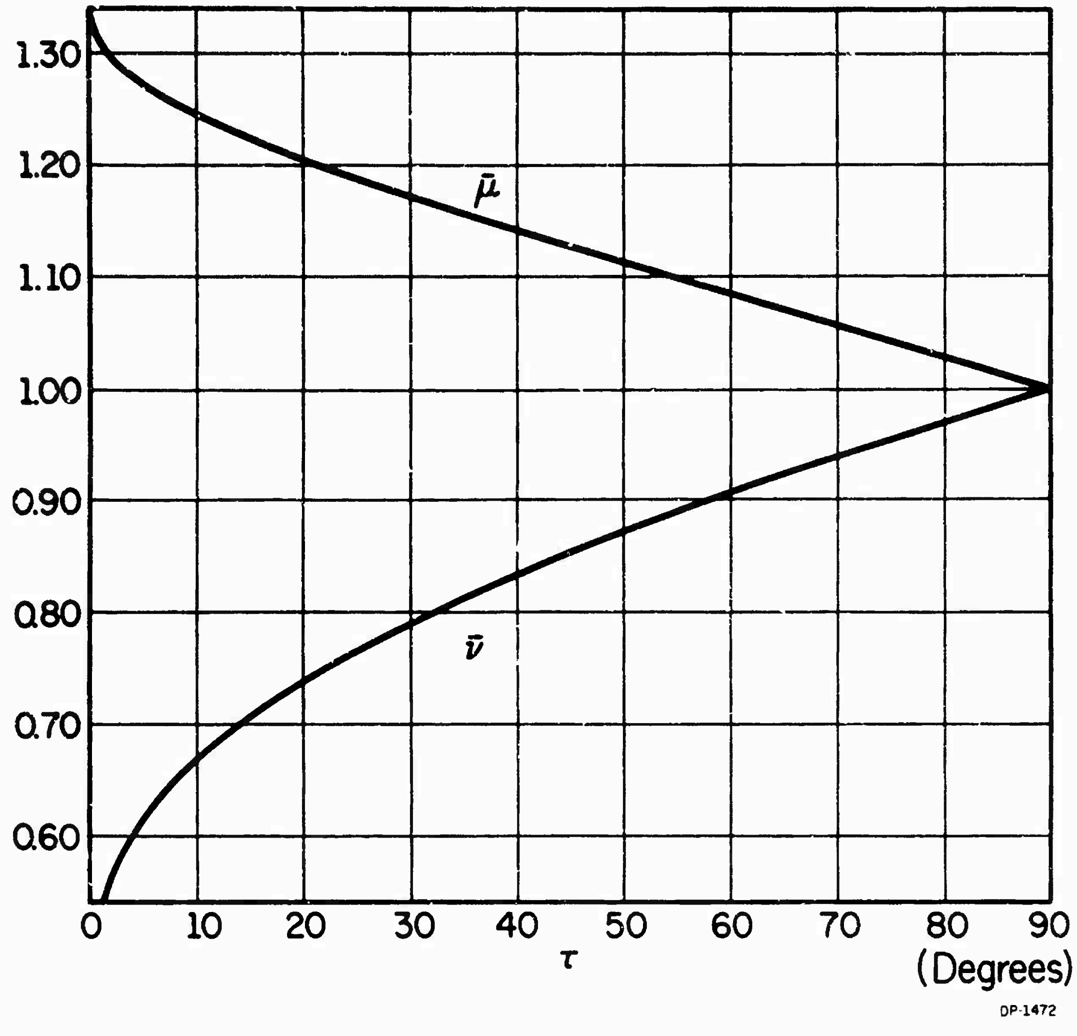

Fig. 6. Plots of Hertzian coefficients in an intermediate reformulation that compares the dimensions of the interface ellipse with those of the interpenetration ellipse for an interpenetration distance that is half the overall deformation, showing the interface ellipse to be, as Hertz observed, somewhat the longer and the narrower of the two.

This substitution provides for writing Eq. (4a) as

$$\gamma_3\delta = (F\gamma_3{}^2/H)^{2/3}\lambda^* \tag{23a}$$

in which

$$\lambda^* = (\lambda/\lambda_o)\lambda_o{}^{1/3}. \tag{24a}$$

With this reformulation, the remaining ones of Eqs. (4) may be written

$$\gamma_3 a = (F\gamma_3{}^2/H)^{1/3}[\gamma_3/(\gamma_5-\gamma_4)]^{\frac{1}{2}}\mu^*, \tag{23b}$$

$$\gamma_3 b = (F\gamma_3{}^2/H)^{1/3}[\gamma_3/(\gamma_5+\gamma_4)]^{\frac{1}{2}}\nu^*, \tag{23c}$$

in which

$$\mu^* = \bar{\mu}\sqrt{\lambda^*}, \tag{24b}$$

$$\nu^* = \bar{\nu}\sqrt{\lambda^*}, \tag{24c}$$

to complete the reformulation

The reformulation represented by Eqs. (23) and (24) is that to be chosen for tabulation. That the range of variation has been substantially reduced may be seen from Figs. 7 and 8, showing λ^*, μ^*, ν^* to both linear and logarithmic scales. Also shown are curves of the geometric mean $\sqrt{(\mu^*\nu^*)}$, which may be used to define the radius of an equivalent circle of contact for the computation of the average interface pressure. The formula is

$$P_{av}/H = (F\gamma_3{}^2/H)^{1/3}/\mu^*\nu^*\pi. \tag{25}$$

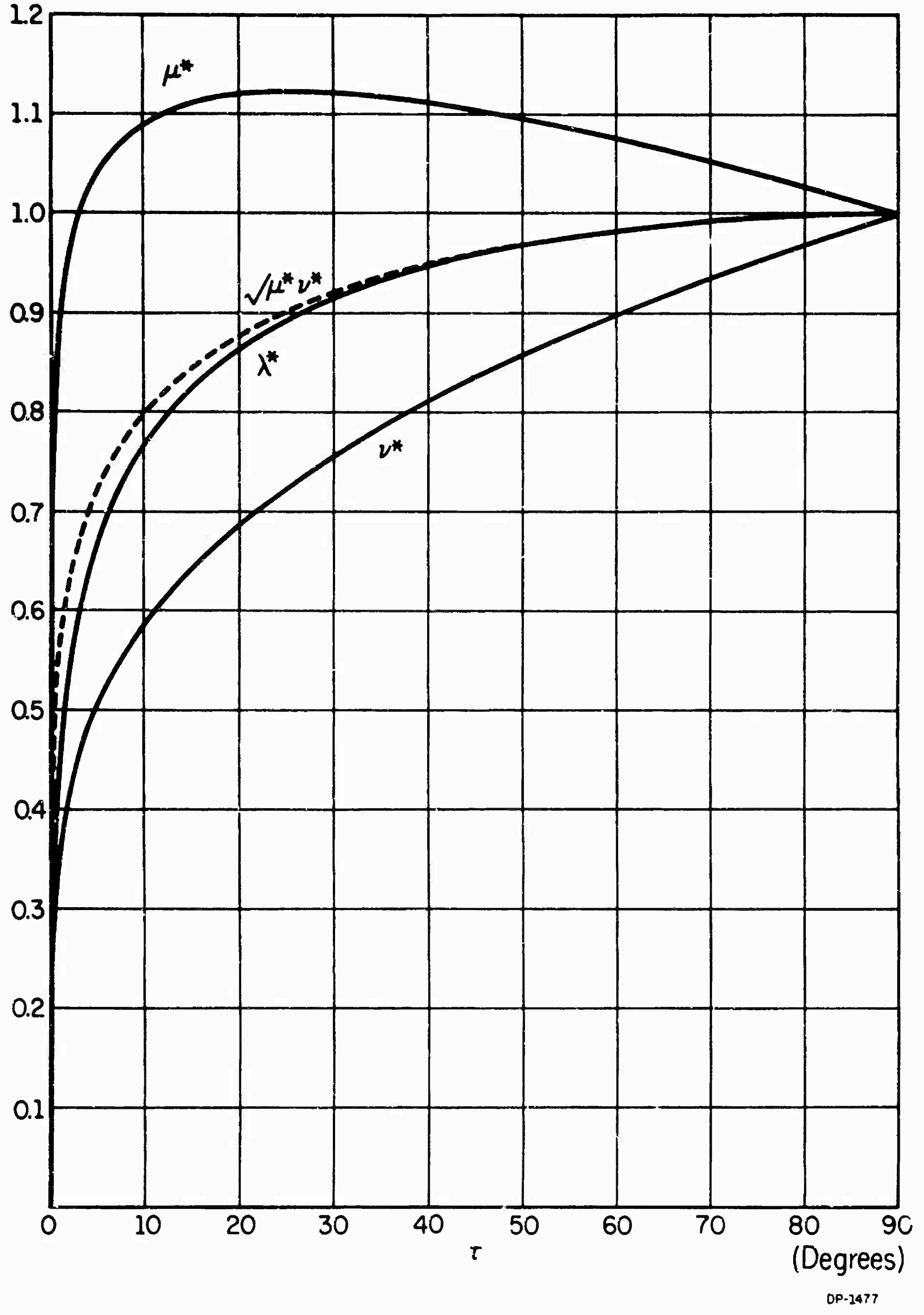

Fig. 7. Plots of Hertzian coefficients in the final reformulation denoted by λ^*, μ^*, ν^*, vs the auxiliary angle τ. These coefficients define the overall deformation and the dimensions of the interface ellipse via Eqs. (23), in their dependence on the undeformed-body curvature parameters of Eqs. (1). Also shown is the geometric mean of the pair of coefficients used to compute interface pressure via Eq. (25).

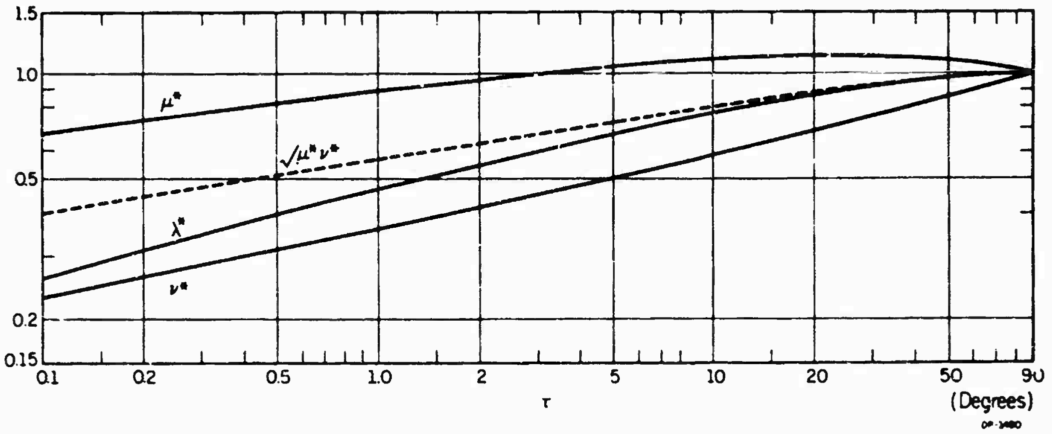

Fig. 8. Plots of the same coefficients as in Fig. 7, except to logarithmic scales. Comparison of these two plots with Figs. 2 and 3 shows that the slighter part of the curvature-parameter dependence is expressed by the reformulated coefficients.

The plot of $\sqrt{(\mu^* \nu^*)}$ shows that it is remarkably close to the value of λ^* over most of the range of variation in τ. Tables from which Figs. 7 and 8 were prepared are given in the appendix.

Many practical cases obtain in which $\omega = 0$. For these cases the formulae, Eqs. (23), are simplified because γ_3, γ_4, and γ_5 obey simpler formulae. These are

$$\gamma_{30} = [(\gamma_{11}+\gamma_{21})(\gamma_{12}+\gamma_{22})]^{\frac{1}{2}}, \tag{26a}$$

$$\gamma_{40} = \tfrac{1}{2}(\gamma_{11}+\gamma_{21})-\tfrac{1}{2}(\gamma_{12}+\gamma_{22}), \tag{26b}$$

$$\gamma_{50} = \tfrac{1}{2}(\gamma_{11}+\gamma_{21})+\tfrac{1}{2}(\gamma_{12}+\gamma_{22}). \tag{26c}$$

Many practical cases also obtain in which the auxiliary angle τ has no simple geometric significance (the exception is the contact between identical cylinders). It is then more convenient to use

$$t = \gamma_4/\gamma_5 = \cos\tau \tag{27}$$

as the auxiliary variable. Also, it is rare that one would be interested in extremely elongated interface ellipses that would obtain for t near unity (τ near 0), except in the limiting case of contact between parallel-axis cylinders for which a separate treatment may be given [3,4]. For this reason, the tabulation given here as Table II presents λ^*, μ^*, and ν^* as functions of the argument t. From the plot shown in Fig. 9, it is seen that the range of variation is quite small in comparison to that of Fig. 2, and that the curvatures are quite small also, for the range of values of t most likely to be of interest. It is Table II, then, that is

Table II. Values of the reformulated Hertzian coefficients, λ^*, μ^*, ν^*, as functions of $t = \cos\tau$ for use in Eqs. (23) and (25), with auxiliary angle τ as defined by Eqs. (1). For plot, see Fig. 9.

t	λ*	μ*	ν*	t	λ*	μ*	ν*
0.00	1.000000	1.000000	1.000000	.50	.983642	1.076128	.899574
.01	.999994	1.001662	.998329	.51	.982853	1.077548	.896993
.02	.999978	1.003317	.996650	.52	.982036	1.078964	.894375
.03	.999950	1.004963	.994962	.53	.981189	1.080377	.891719
.04	.999911	1.006602	.993264	.54	.980311	1.081787	.889021
.05	.999861	1.008233	.991558	.55	.979401	1.083193	.886281
.06	.999800	1.009857	.989842	.56	.978457	1.084595	.883498
.07	.999727	1.011474	.988117	.57	.977478	1.085993	.880668
.08	.999643	1.013084	.986381	.58	.976463	1.087386	.877790
.09	.999548	1.014686	.984636	.59	.975409	1.088776	.874863
.10	.999441	1.016283	.982879	.60	.974314	1.090161	.871882
.11	.999323	1.017872	.981113	.61	.973177	1.091540	.868847
.12	.999193	1.019456	.979335	.62	.971996	1.092915	.865753
.13	.999052	1.021033	.977546	.63	.970767	1.094284	.862599
.14	.998898	1.022604	.975745	.64	.969489	1.095647	.859382
.15	.998733	1.024169	.973932	.65	.968158	1.097003	.856097
.16	.998556	1.025729	.972107	.66	.966772	1.098352	.852741
.17	.998367	1.027283	.970270	.67	.965327	1.099693	.849311
.18	.998165	1.028831	.968420	.68	.963819	1.101026	.845801
.19	.997951	1.030374	.966556	.69	.962245	1.102350	.842208
.20	.997724	1.031912	.964679	.70	.960599	1.103663	.838526
.21	.997485	1.033445	.962788	.71	.958877	1.104965	.834750
.22	.997233	1.034973	.960883	.72	.957074	1.106254	.830872
.23	.996967	1.036496	.958963	.73	.955183	1.107530	.826888
.24	.996688	1.038014	.957028	.74	.953198	1.108789	.822789
.25	.996395	1.039528	.955077	.75	.951110	1.110031	.818566
.26	.996089	1.041038	.953110	.76	.948912	1.111253	.814211
.27	.995768	1.042543	.951127	.77	.946593	1.112453	.809712
.28	.995434	1.044043	.949126	.78	.944143	1.113627	.805059
.29	.995084	1.045540	.947109	.79	.941549	1.114772	.800238
.30	.994720	1.047032	.945073	.80	.938797	1.115884	.795234
.31	.994340	1.048520	.943019	.81	.935869	1.116957	.790029
.32	.993945	1.050005	.940946	.82	.932747	1.117985	.784604
.33	.993533	1.051486	.938853	.83	.929407	1.118961	.778934
.34	.993106	1.052962	.936740	.84	.925822	1.119877	.772994
.35	.992662	1.054436	.934606	.85	.921961	1.120720	.766751
.36	.992200	1.055905	.932451	.86	.917783	1.121478	.760166
.37	.991722	1.057371	.930273	.87	.913240	1.122132	.753193
.38	.991225	1.058834	.928073	.88	.908273	1.122661	.745775
.39	.990709	1.060293	.925849	.89	.902807	1.123035	.737840
.40	.990175	1.061749	.923601	.90	.896744	1.123215	.729298
.41	.989621	1.063202	.921327	.91	.889956	1.123150	.720032
.42	.989046	1.064651	.919027	.92	.882268	1.122765	.709885
.43	.988451	1.066097	.916701	.93	.873439	1.121953	.698641
.44	.987835	1.067539	.914346	.94	.863109	1.120552	.685990
.45	.987196	1.068979	.911963	.95	.850727	1.118301	.671460
.46	.986534	1.070415	.909549	.96	.835369	1.114743	.654282
.47	.985849	1.071848	.907105	.97	.815309	1.108989	.633057
.48	.985139	1.073278	.904628	.98	.786702	1.098938	.604763
.49	.984404	1.074705	.902118	.99	.737476	1.077506	.560295
.50	.983642	1.076128	.899574	1.00	0.000000	0.000000	0.000000

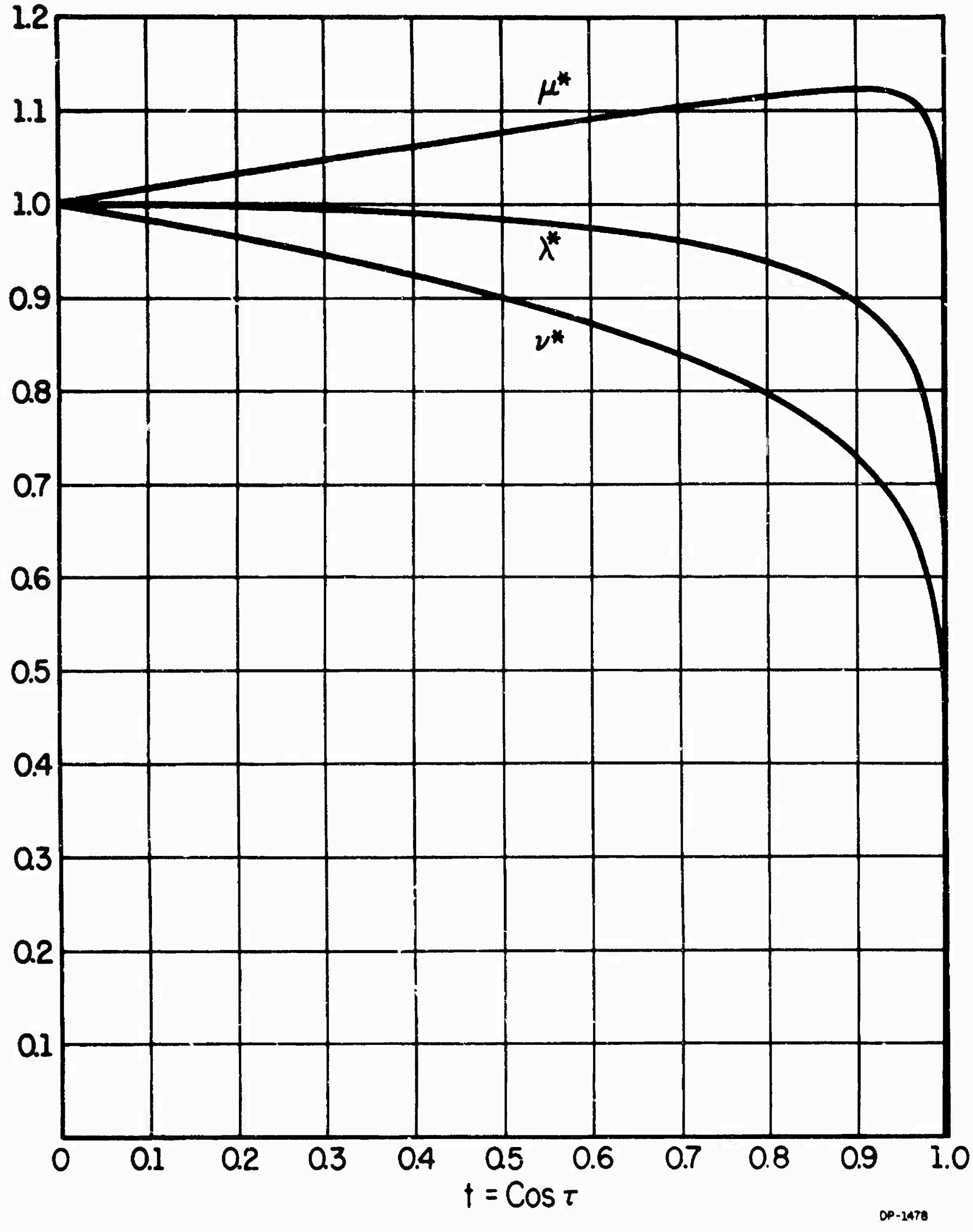

Fig. 9. Plots of the reformulated Hertzian coefficients λ^*, μ^*, ν^*, vs $t=\cos\tau$, as tabulated in Table II. For the greater variety of practical cases, t will not be close to unity (τ will not be close to 0), so that the straightne-- of the curve indicates that interpolation in Table II need involve few precautions.

thought most likely to be of wide utility.

Table II and the plot of Fig. 9 may be extended to negative values of t by the formulae,

$$\lambda^*(-t) = \lambda^*(t), \tag{28a}$$

$$\mu^*(-t) = \nu^*(t), \tag{28b}$$

$$\nu^*(-t) = \mu^*(t). \tag{28c}$$

Similar formulae extend the plots <u>versus</u> τ beyond 90°. These are

$$\lambda(\tau) = \lambda(180^\circ-\tau), \tag{29a}$$

$$\mu(\tau) = \nu(180^\circ-\tau), \tag{29b}$$

$$\nu(\tau) = \mu(180^\circ-\tau), \tag{29c}$$

applicable also to the various reformulations denoted by bars and stars.

Acknowledgment

Thanks are due to Mr. K. C. Kelley for programming the computer to produce the console-typewriter print photographically reproduced here as Table II.

Appendix: Supplementary Tables of Hertzian Coefficients

Although the original table of Hertz with the corrections shown above in Table I, the values shown graphically in Figs. 2, 3, 4, 6, 8, and 9, together with values that may be computed by hand methods, will satisfy needs for values of the Hertzian coefficients of moderate accuracy, and the

values shown above in Table II will satisfy needs for high accuracy in most cases, there are some few specialized interests that may remain unsatisfied.

Very accurate values for the original formulation of these coefficients as used by Hertz may still be of interest to some. For them, the six-place tabulation of Table AI is presented for increments in the auxiliary angle of 1°. For those whose interest extends to extremely elongated interface ellipses involving extremely small values of the auxiliary angle, the logarithmic tabulation of Table AII extends the range down to 0.01°.

Others may find it convenient to work with the coefficients in the reformulation denoted by stars, but, because of an interest in extremely elongated interface ellipses, find the argument represented by the cosine of the auxiliary angle, as used above in Table II, inconvenient. For them, the six-place tabulation of Table AIII is presented for increments in the auxiliary angle of 1°. When the interest extends to angles less than 10°, the logarithmic tabulation of Table AIV extending the range down to 0.01° should be used.

Table AI. Values of the Hertzian coefficients λ, $1/\mu$, ν versus τ, the auxiliary angle in degrees, covering $0^{\circ} \leq \tau \leq 90^{\circ}$.

τ	λ	$1/\mu$	ν	τ	λ	$1/\mu$	ν
0	0.000000	0.000000	0.000000	45	.854714	.519086	.603828
1	.121107	.027077	.131295	46	.861499	.529269	.611159
2	.179314	.044958	.169191	47	.868091	.539458	.618507
3	.224669	.060689	.196597	48	.874493	.549655	.625875
4	.263071	.075231	.218919	49	.880709	.559861	.633267
5	.296894	.088983	.238136	50	.886742	.570077	.640685
6	.327391	.102163	.255224	51	.892594	.580304	.648135
7	.355319	.114902	.270746	52	.898269	.590544	.655618
8	.381179	.127291	.285061	53	.903770	.600796	.663138
9	.405324	.139392	.298415	54	.909099	.611063	.670699
10	.428013	.151253	.310983	55	.914259	.621346	.678303
11	.449443	.162910	.322896	56	.919251	.631645	.685955
12	.469770	.174392	.334254	57	.924079	.641962	.693658
13	.489118	.185722	.345135	58	.928744	.652297	.701414
14	.507586	.196920	.355604	59	.933247	.662652	.709227
15	.525261	.208000	.365713	60	.937592	.673027	.717100
16	.542210	.218977	.375504	61	.941780	.683425	.725037
17	.558495	.229861	.385015	62	.945812	.693845	.733041
18	.574167	.240663	.394275	63	.949690	.704289	.741116
19	.589269	.251392	.403312	64	.953416	.714759	.749264
20	.603842	.262054	.412150	65	.956991	.725255	.757490
21	.617917	.272656	.420807	66	.960415	.735778	.765797
22	.631527	.283206	.429303	67	.963692	.746329	.774188
23	.644696	.293706	.437653	68	.966820	.756911	.782667
24	.657450	.304164	.445871	69	.969803	.767523	.791238
25	.669810	.314582	.453971	70	.972640	.778167	.799904
26	.681795	.324965	.461963	71	.975333	.788845	.808670
27	.693422	.335317	.469859	72	.977883	.799557	.817540
28	.704707	.345640	.477668	73	.980290	.810306	.826516
29	.715665	.355938	.485399	74	.982556	.821091	.835605
30	.726310	.366213	.493060	75	.984681	.831916	.844808
31	.736652	.376469	.500659	76	.986665	.842780	.854132
32	.746704	.386707	.508202	77	.988510	.853686	.863580
33	.756475	.396930	.515698	78	.990216	.864635	.873157
34	.765975	.407140	.523151	79	.991784	.875629	.882868
35	.775213	.417339	.530567	80	.993213	.886669	.892717
36	.784197	.427528	.537953	81	.994506	.897757	.902710
37	.792935	.437710	.545313	82	.995661	.908894	.912851
38	.801433	.447887	.552653	83	.996679	.920083	.923146
39	.809698	.458059	.559976	84	.997561	.931325	.933601
40	.817736	.468229	.567289	85	.998307	.942621	.944220
41	.825554	.478397	.574594	86	.998916	.953975	.955010
42	.833155	.488566	.581897	87	.999391	.965388	.965976
43	.840546	.498736	.589200	88	.999729	.976861	.977126
44	.847731	.508909	.596510	89	.999932	.988398	.988465
45	.854714	.519086	.603828	90	1.000000	1.000000	1.000000

Table AII. Values of common logarithms of the Hertzian coefficients $\Lambda = 3+\log\lambda$, $M = \log\mu$, $N = 2+\log\nu$, versus $T = 2+\log\tau$, for the auxiliary angle τ in degrees, covering the range $0.01^\circ \leq \tau \leq 10^\circ$.

T	Λ	M	N
0.00	.893551	2.986916	.408482
.10	.954572	2.917015	.443433
.20	1.015474	2.847037	.478422
.30	1.076254	2.776977	.513452
.40	1.136905	2.706832	.548524
.50	1.197421	2.636596	.583642
.60	1.257796	2.566265	.618808
.70	1.318024	2.495833	.654024
.80	1.378097	2.425295	.689293
.90	1.438006	2.354644	.724618
1.00	1.497744	2.283873	.760003
1.10	1.557302	2.212975	.795453
1.20	1.616668	2.141941	.830970
1.30	1.675833	2.070762	.866560
1.40	1.734784	1.999427	.902227
1.50	1.793508	1.927925	.937978
1.60	1.851991	1.856243	.973820
1.70	1.910217	1.784368	1.009759
1.80	1.968168	1.712281	1.045803
1.90	2.025826	1.639967	1.081962
2.00	2.083168	1.567402	1.118248
2.10	2.140169	1.494564	1.154672
2.20	2.196802	1.421426	1.191250
2.30	2.253037	1.347955	1.227998
2.40	2.308835	1.274116	1.264940
2.50	2.364156	1.199864	1.302100
2.60	2.418950	1.125150	1.339511
2.70	2.473157	1.049914	1.377215
2.80	2.526707	.974085	1.415268
2.90	2.579511	.897580	1.453742
3.00	2.631457	.820297	1.492737

Table AIII. Values of the reformulated Hertzian coefficients λ^*, μ^*, ν^*, versus τ, the auxiliary angle in degrees, covering $0^\circ \leq \tau \leq 90^\circ$.

τ	λ^*	μ^*	ν^*	τ	λ^*	μ^*	ν^*
0	0.000000	0.000000	0.000000	45	.959384	1.104590	.835852
1	.466891	.894918	.364561	46	.961487	1.102963	.840504
2	.548707	.960352	.418495	47	.963510	1.101291	.845090
3	.600635	.997373	.454440	48	.965455	1.099577	.849612
4	.639064	1.022528	.482248	49	.967325	1.097821	.854072
5	.669633	1.041126	.505294	50	.969124	1.096025	.858473
6	.695007	1.055561	.525172	51	.970852	1.094191	.862816
7	.716673	1.067120	.542771	52	.972514	1.092319	.867103
8	.735548	1.076575	.558644	53	.974111	1.090411	.871335
9	.752243	1.084425	.573157	54	.975645	1.088468	.875515
10	.767185	1.091012	.586568	55	.977119	1.086492	.879645
11	.780684	1.096581	.599065	56	.978534	1.084482	.883724
12	.792974	1.101310	.610791	57	.979893	1.082440	.887756
13	.804237	1.105337	.621855	58	.981196	1.080366	.891740
14	.814614	1.108769	.632344	59	.982445	1.078262	.895679
15	.824219	1.111689	.642329	60	.983642	1.076128	.899574
16	.833147	1.114165	.651868	61	.984789	1.073965	.903425
17	.841475	1.116252	.661007	62	.985886	1.071773	.907235
18	.849267	1.117997	.669788	63	.986935	1.069552	.911003
19	.856577	1.119438	.678243	64	.987937	1.067305	.914732
20	.863454	1.120606	.686404	65	.988893	1.065030	.918421
21	.869936	1.121530	.694293	66	.989804	1.062728	.922072
22	.876059	1.122233	.701934	67	.990671	1.060400	.925686
23	.881852	1.122735	.709346	68	.991495	1.058046	.929263
24	.887344	1.123054	.716545	69	.992277	1.055666	.932804
25	.892556	1.123207	.723546	70	.993018	1.053260	.936311
26	.897511	1.123205	.730364	71	.993718	1.050830	.939783
27	.902226	1.123062	.737009	72	.994378	1.048374	.943222
28	.906718	1.122789	.743493	73	.994999	1.045894	.946628
29	.911002	1.122393	.749825	74	.995581	1.043389	.950001
30	.915093	1.121886	.756014	75	.996126	1.040860	.953343
31	.919001	1.121273	.762069	76	.996633	1.038306	.956654
32	.922738	1.120562	.767995	77	.997103	1.035728	.959934
33	.926314	1.119759	.773801	78	.997536	1.033125	.963184
34	.929738	1.118870	.779491	79	.997933	1.030499	.966405
35	.933019	1.117900	.785073	80	.998295	1.027848	.969597
36	.936165	1.116853	.790550	81	.998621	1.025173	.972760
37	.939182	1.115734	.795928	82	.998912	1.022474	.975894
38	.942077	1.114547	.801211	83	.999168	1.019751	.979001
39	.944856	1.113295	.806404	84	.999389	1.017003	.982081
40	.947525	1.111981	.811510	85	.999576	1.014231	.985133
41	.950089	1.110610	.816532	86	.999729	1.011435	.988159
42	.952552	1.109182	.821474	87	.999848	1.008613	.991158
43	.954920	1.107701	.826340	88	.999932	1.005767	.994131
44	.957196	1.106170	.831132	89	.999983	1.002896	.997078
45	.959384	1.104590	.835852	90	1.000000	1.000000	1.000000

Table AIV. Values of common logarithms of the reformulated Hertzian coefficients $\Lambda^* = 1+\log\lambda^*$, $M^* = 1+\log\mu^*$, $N^* = 1+\log\nu^*$, versus $T = 2+\log\tau$, for the auxiliary angle τ in degrees, covering the range $0.01^\circ \leq \tau \leq 10^\circ$.

T	Λ^*	M*	N*
0.00	.146259	.704632	.185351
.10	.173946	.718065	.203635
.20	.201515	.731420	.221957
.30	.228961	.744693	.240320
.40	.256279	.757981	.258726
.50	.283462	.770979	.277178
.60	.310504	.783981	.295676
.70	.337398	.796883	.314226
.80	.364137	.809678	.332828
.90	.390714	.822360	.351487
1.00	.417119	.834923	.370205
1.10	.443343	.847358	.388988
1.20	.469376	.859657	.407838
1.30	.495207	.871811	.426761
1.40	.520825	.883810	.445762
1.50	.546216	.895641	.464846
1.60	.571366	.907293	.484020
1.70	.596260	.918750	.503291
1.80	.620879	.929997	.522667
1.90	.645205	.941015	.542156
2.00	.669216	.951783	.561771
2.10	.692888	.962278	.581520
2.20	.716195	.972471	.601420
2.30	.739107	.982331	.621483
2.40	.761589	.991820	.641727
2.50	.783604	1.000895	.662173
2.60	.805107	1.009503	.682842
2.70	.826049	1.017584	.703761
2.80	.846374	1.025061	.724958
2.90	.866016	1.031847	.746465
3.00	.884900	1.037830	.768318

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2. The notation for these coefficients is not to be confused with that which is sometimes used for certain combinations of elastic moduli.

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13 ABSTRACT

Formulations are given for the coefficients , , , defined by H. Hertz in terms of the solution of a transcendental equation involving elliptic integrals and used by him to describe the deformation of bodies subjected to contact stresses. Methods of approximate calculation are explained, errors in the tables prepared by Hertz are noted. For the purpose of providing a more extensive and more accurate tabulation, using an automatic digital computer, these coefficients are reformulated so that a large part of the variation is expressed by means of easily-interpreted elementary formulae. The remainder of the variation is tabulated to 6 places for 100 values of the argument. Graphs of the coefficients are also provided.

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Hertz						
Contact Stress						
Deformation						
Elasticity						
Table						
Error						

Security Classification